A New Algebraic Structure of Finite Quantum Systems and the Modified Bessel Functions

Kazuyuki FUJII *

Department of Mathematical Sciences

Yokohama City University

Yokohama, 236-0027

Japan

Abstract

In this paper we present a new algebraic structure (a super hyperbolic system in our terminology) for finite quantum systems, which is a generalization of the usual one in the two–level system.

It fits into the so–called generalized Pauli matrices, so they play an important role in the theory. Some deep relation to the modified Bessel functions of integer order is pointed out.

By taking a skillful limit finite quantum systems become quantum mechanics on the circle developed by Ohnuki and Kitakado.

*E-mail address : fujii@yokohama-cu.ac.jp

Quantum Computation is usually based on two–level system of atoms (qubit theory). In the realistic construction of quantum logic gates we must solve some Schrödinger equations. Then the Pauli matrices $\{\sigma_1, \sigma_3\}$ is essentially used and not only the periodic functions $\{\cos(x), \sin(x)\}$ but also the hyperbolic functions $\{\cosh(x), \sinh(x)\}$ play an important role.

On the other hand, they are deeply related to the modified Bessel functions of integer order $\{I_n(x) \mid n \in \mathbf{Z}\}$. The functions are in general given by the generating function.

Atom has usually many (finite or infinite) energy levels. However, to treat infinitely many ones at the same time is not realistic, so we treat an atom with finite (for example n) energy levels. We call this a finite quantum system and for this system the so–called generalized Pauli matrices $\{\Sigma_1, \Sigma_3\}$ play a crucial role, see for example [1], [2] and [3].

In this system we have a natural question on what functions corresponding to the hyperbolic functions are. In the paper we present such a system $\{c_0(x), c_1(x), \cdots, c_{n-1}(x)\}$ (a super hyperbolic system in our terminology) as a "natural" generalization of $\{\cosh(x), \sinh(x)\}$.

Moreover, we define a generating matrix based on the generalized Pauli matrices as a "natural" generalization of the generating function and obtain interesting results by taking some traces.

Lastly, we want to take a limit of finite quantum systems, which is of course impossible. However, there is a bypass. That is, by taking a skillful limit finite quantum systems become quantum mechanics on the circle developed by Ohnuki and Kitakado [4].

Through this paper we have a clear and unified picture of quantum systems.

First of all we make some mathematical preliminaries on the 2–level system. Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be Pauli matrices and $\mathbf{1}_2$ the unit matrix :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

List the well–known properties of σ_1 and σ_3 :

$$\sigma_1^2 = \sigma_3^2 = \mathbf{1}_2, \quad \sigma_1^\dagger = \sigma_1, \quad \sigma_3^\dagger = \sigma_3, \quad \sigma_3\sigma_1 = -\sigma_1\sigma_3 = \mathrm{e}^{\pi i}\sigma_1\sigma_3. \tag{2}$$

Let W be the Walsh–Hadamard matrix

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W^{-1}\ , \tag{3}$$

then we can diagonalize σ_1 as $\sigma_1 = W\sigma_3 W^{-1}$ by making use of W.

The modified Bessel functions of integer order $\{I_k(x) \mid k \in \mathbf{Z}\}$ are given by the generating function

$$\mathrm{e}^{\frac{x}{2}(w+\frac{1}{w})} = \sum_{k\in\mathbf{Z}} I_k(x) w^k. \tag{4}$$

Now let us list some (well–known) important properties (see for example [5]) :

$$\begin{aligned} 1 &= I_0(x) + 2\sum_{k=1}^{\infty}(-1)^k I_{2k}(x), \\ \mathrm{e}^x &= I_0(x) + 2\sum_{k=1}^{\infty} I_k(x), \quad \mathrm{e}^{-x} = I_0(x) + 2\sum_{k=1}^{\infty}(-1)^k I_k(x) \\ \cosh(x) &= I_0(x) + 2\sum_{k=1}^{\infty} I_{2k}(x), \quad \sinh(x) = 2\sum_{k=1}^{\infty} I_{2k-1}(x). \end{aligned}$$

In the following we set

$$c_0(x) \equiv \cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad c_1(x) \equiv \sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \tag{5}$$

for simplicity. The fundamental equation

$$c_0^2(x) - c_1^2(x) = 1 \tag{6}$$

is interpreted as a simple relation

$$S\sigma_3 S = \sigma_3 \iff \sigma_3 S\sigma_3 S = \mathbf{1}_2 \iff (\sigma_3 S)^2 = \mathbf{1}_2$$

for S defined by

$$S = \begin{pmatrix} c_0(x) & c_1(x) \\ c_1(x) & c_0(x) \end{pmatrix} = c_0(x)\mathbf{1}_2 + c_1(x)\sigma_1 = \mathrm{e}^{x\sigma_1}. \tag{7}$$

Next we would like to extend the 2–level system to general n–level one. To make our purpose clearer we treat the 3–level case in detail. Let σ be $\exp(\frac{2\pi i}{3})$, then we have

$$\sigma^3 = 1, \quad \bar{\sigma} = \sigma^2, \quad 1 + \sigma + \sigma^2 = 0. \tag{8}$$

Let Σ_1 and Σ_3 be generators of generalized Pauli matrices in the case of $n = 3$, namely

$$\Sigma_1 = \begin{pmatrix} 0 & & 1 \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1 & & \\ & \sigma & \\ & & \sigma^2 \end{pmatrix}. \tag{9}$$

Then it is easy to see

$$\Sigma_1^3 = \Sigma_3^3 = \mathbf{1}_3, \quad \Sigma_1^\dagger = \Sigma_1^2, \quad \Sigma_3^\dagger = \Sigma_3^2, \quad \Sigma_3\Sigma_1 = \sigma\Sigma_1\Sigma_3. \tag{10}$$

Now we can show that Σ_1 can be diagonalized by making use of the matrix

$$W = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 \end{pmatrix} \quad \in \quad U(3) \tag{11}$$

like

$$\Sigma_1 = W\Sigma_3 W^\dagger = W\Sigma_3 W^{-1}. \tag{12}$$

In fact

$$W\Sigma_3 W^\dagger = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \sigma & \\ & & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma & \sigma^2 \\ 1 & \sigma^2 & \sigma \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \Sigma_1,$$

where we have used the relations in (8).

From (5) we set

$$c_0(x) = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!}, \quad c_1(x) = \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!}, \quad c_2(x) = \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}. \tag{13}$$

Then it is easy to check

$$c_0(x) = \frac{\mathrm{e}^x + \mathrm{e}^{\sigma x} + \mathrm{e}^{\sigma^2 x}}{3}, \quad c_1(x) = \frac{\mathrm{e}^x + \sigma^2\mathrm{e}^{\sigma x} + \sigma\mathrm{e}^{\sigma^2 x}}{3}, \quad c_2(x) = \frac{\mathrm{e}^x + \sigma\mathrm{e}^{\sigma x} + \sigma^2\mathrm{e}^{\sigma^2 x}}{3} \tag{14}$$

by use of σ in (8) or reversely

$$\mathrm{e}^x = c_0(x)+c_1(x)+c_2(x), \quad \mathrm{e}^{\sigma x} = c_0(x)+\sigma c_1(x)+\sigma^2 c_2(x), \quad \mathrm{e}^{\sigma^2 x} = c_0(x)+\sigma^2 c_1(x)+\sigma c_2(x).$$

Now, our question is as follows : What is the fundamental equation that $\{c_0(x), c_1(x), c_2(x)\}$ satisfy ?

The answer is given by the equation

$$
\begin{aligned}
&(c_0(x)+c_1(x)+c_2(x))(c_0(x)+\sigma c_1(x)+\sigma^2 c_2(x))(c_0(x)+\sigma^2 c_1(x)+\sigma c_2(x)) \\
&= \mathrm{e}^{x}\mathrm{e}^{\sigma x}\mathrm{e}^{\sigma^2 x} = \mathrm{e}^{(1+\sigma+\sigma^2)x} = \mathrm{e}^{0} = 1.
\end{aligned}
$$

By expanding the left-hand side and using the relations (8) we obtain

$$
c_0^3(x)+c_1^3(x)+c_2^3(x)-3c_0(x)c_1(x)c_2(x)=1. \tag{15}
$$

Next let us consider the addition formulas. By expanding

$$
\mathrm{e}^{\sigma x}\mathrm{e}^{\sigma y}=\mathrm{e}^{\sigma(x+y)} \quad \Longleftarrow \quad \mathrm{e}^{\sigma t}=c_0(t)+\sigma c_1(t)+\sigma^2 c_2(t)
$$

we have

$$
\begin{aligned}
c_0(x)c_0(y)+c_1(x)c_2(y)+c_2(x)c_1(y)&=c_0(x+y),\\
c_0(x)c_1(y)+c_1(x)c_0(y)+c_2(x)c_2(y)&=c_1(x+y),\\
c_0(x)c_2(y)+c_1(x)c_1(y)+c_2(x)c_0(y)&=c_2(x+y).
\end{aligned} \tag{16}
$$

From here let us give a unified approach by use of the generalized Pauli matrices $\{\Sigma_1, \Sigma_3\}$ above. We consider the matrix

$$
\mathrm{e}^{x\Sigma_1}=c_0(x)\mathbf{1}_3+c_1(x)\Sigma_1+c_2(x)\Sigma_1^2=
\begin{pmatrix}
c_0(x) & c_2(x) & c_1(x)\\
c_1(x) & c_0(x) & c_2(x)\\
c_2(x) & c_1(x) & c_0(x)
\end{pmatrix}. \tag{17}
$$

Then by $\Sigma_1 = W\Sigma_3 W^{\dagger}$ in (12)

$$
\begin{pmatrix}
c_0(x) & c_2(x) & c_1(x)\\
c_1(x) & c_0(x) & c_2(x)\\
c_2(x) & c_1(x) & c_0(x)
\end{pmatrix}
=\mathrm{e}^{x\Sigma_1}=W\mathrm{e}^{x\Sigma_3}W^{\dagger}=W
\begin{pmatrix}
\mathrm{e}^{x} & & \\
 & \mathrm{e}^{x\sigma} & \\
 & & \mathrm{e}^{x\sigma^2}
\end{pmatrix}W^{\dagger},
$$

so taking the determinant leads to

$$\begin{vmatrix} c_0(x) & c_2(x) & c_1(x) \\ c_1(x) & c_0(x) & c_2(x) \\ c_2(x) & c_1(x) & c_0(x) \end{vmatrix} = \mathrm{e}^{(1+\sigma+\sigma^2)x} = 1.$$

Namely, we recovered (15).

On the other hand, by use of (11) it is straightforward to show

$$W \begin{pmatrix} \mathrm{e}^{x} & & \\ & \mathrm{e}^{x\sigma} & \\ & & \mathrm{e}^{x\sigma^2} \end{pmatrix} W^\dagger = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{x} & & \\ & \mathrm{e}^{x\sigma} & \\ & & \mathrm{e}^{x\sigma^2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma & \sigma^2 \\ 1 & \sigma^2 & \sigma \end{pmatrix}$$
$$= \frac{\mathrm{e}^{x} + \mathrm{e}^{\sigma x} + \mathrm{e}^{\sigma^2 x}}{3}\mathbf{1}_3 + \frac{\mathrm{e}^{x} + \sigma^2\mathrm{e}^{\sigma x} + \sigma\mathrm{e}^{\sigma^2 x}}{3}\Sigma_1 + \frac{\mathrm{e}^{x} + \sigma\mathrm{e}^{\sigma x} + \sigma^2\mathrm{e}^{\sigma^2 x}}{3}\Sigma_1^2,$$

so we recovered (14).

The matrix form is very convenient. Moreover, we can give new relations. For that we consider the simple equation

$$\mathrm{e}^{x\Sigma_1}\mathrm{e}^{y\Sigma_1^\dagger} = \mathrm{e}^{x\Sigma_1 + y\Sigma_1^\dagger}. \tag{18}$$

The left hand side is

$$\begin{aligned} \mathrm{e}^{x\Sigma_1}\mathrm{e}^{y\Sigma_1^\dagger} &= (c_0(x)\mathbf{1}_3 + c_1(x)\Sigma_1 + c_2(x)\Sigma_1^2)(c_0(y)\mathbf{1}_3 + c_1(y)\Sigma_1^2 + c_2(y)\Sigma_1) \\ &= (c_0(x)c_0(y) + c_1(x)c_1(y) + c_2(x)c_2(y))\mathbf{1}_3 \\ &+ (c_0(x)c_2(y) + c_1(x)c_0(y) + c_2(x)c_1(y))\Sigma_1 \\ &+ (c_0(x)c_1(y) + c_1(x)c_2(y) + c_2(x)c_0(y))\Sigma_1^2 \end{aligned}$$

because $\Sigma_1^\dagger = \Sigma_1^2$. The right hand side is

$$\mathrm{e}^{x\Sigma_1 + y\Sigma_1^\dagger} = \mathrm{e}^{W(x\Sigma_3 + y\Sigma_3^2)W^\dagger} = W\mathrm{e}^{x\Sigma_3 + y\Sigma_3^2}W^\dagger = W \begin{pmatrix} \mathrm{e}^{x+y} & & \\ & \mathrm{e}^{x\sigma + y\sigma^2} & \\ & & \mathrm{e}^{x\sigma^2 + y\sigma} \end{pmatrix} W^\dagger$$
$$= \frac{\mathrm{e}^{x+y} + \mathrm{e}^{x\sigma+y\sigma^2} + \mathrm{e}^{x\sigma^2+y\sigma}}{3}\mathbf{1}_3 + \frac{\mathrm{e}^{x+y} + \sigma^2\mathrm{e}^{x\sigma+y\sigma^2} + \sigma\mathrm{e}^{x\sigma^2+y\sigma}}{3}\Sigma_1$$
$$+ \frac{\mathrm{e}^{x+y} + \sigma\mathrm{e}^{x\sigma+y\sigma^2} + \sigma^2\mathrm{e}^{x\sigma^2+y\sigma}}{3}\Sigma_1^2,$$

so we obtain

$$
\begin{aligned}
c_0(x)c_0(y)+c_1(x)c_1(y)+c_2(x)c_2(y) &= \frac{\mathrm{e}^{x+y}+\mathrm{e}^{x\sigma+y\sigma^2}+\mathrm{e}^{x\sigma^2+y\sigma}}{3}, \\
c_0(x)c_2(y)+c_1(x)c_0(y)+c_2(x)c_1(y) &= \frac{\mathrm{e}^{x+y}+\sigma^2\mathrm{e}^{x\sigma+y\sigma^2}+\sigma\mathrm{e}^{x\sigma^2+y\sigma}}{3}, \\
c_0(x)c_1(y)+c_1(x)c_2(y)+c_2(x)c_0(y) &= \frac{\mathrm{e}^{x+y}+\sigma\mathrm{e}^{x\sigma+y\sigma^2}+\sigma^2\mathrm{e}^{x\sigma^2+y\sigma}}{3}.
\end{aligned} \tag{19}
$$

Next, let us consider the matrix $\mathrm{e}^{x\Sigma_1+y\Sigma_1^\dagger}$. If we set $y=1/x$, then the matrix $\mathrm{e}^{x\Sigma_1+(1/x)\Sigma_1^\dagger}$ is similar to (4) the generating function of modified Bessel functions of integer order. Therefore from (4) it is reasonable to consider

$$
\mathrm{e}^{\frac{x}{2}\left(w\Sigma_1+\frac{1}{w}\Sigma_1^\dagger\right)}=\mathrm{e}^{\frac{x}{2}\left(w\Sigma_1+\frac{1}{w}\Sigma_1^{-1}\right)}=\sum_{k\in\mathbf{Z}} I_k(x)w^k\Sigma_1^k. \tag{20}
$$

In the following we call this the **generating matrix** of modified Bessel functions of integer order. Let us look for some typical properties. The result is

$$
\begin{aligned}
\frac{1}{3}\mathrm{tr}\left\{\mathrm{e}^{\frac{x}{2}\left(w\Sigma_1+\frac{1}{w}\Sigma_1^\dagger\right)}\right\} &= \frac{\mathrm{e}^{\frac{x}{2}(w+\frac{1}{w})}+\mathrm{e}^{\frac{x}{2}(w\sigma+\frac{1}{w}\sigma^2)}+\mathrm{e}^{\frac{x}{2}(w\sigma^2+\frac{1}{w}\sigma)}}{3}=\sum_{k\in\mathbf{Z}} I_{3k}(x)w^{3k}, \\
\frac{1}{3}\mathrm{tr}\left\{\mathrm{e}^{\frac{x}{2}\left(w\Sigma_1+\frac{1}{w}\Sigma_1^\dagger\right)}\Sigma_1\right\} &= \frac{\mathrm{e}^{\frac{x}{2}(w+\frac{1}{w})}+\sigma\mathrm{e}^{\frac{x}{2}(w\sigma+\frac{1}{w}\sigma^2)}+\sigma^2\mathrm{e}^{\frac{x}{2}(w\sigma^2+\frac{1}{w}\sigma)}}{3}=\sum_{k\in\mathbf{Z}} I_{3k-1}(x)w^{3k-1}, \\
\frac{1}{3}\mathrm{tr}\left\{\mathrm{e}^{\frac{x}{2}\left(w\Sigma_1+\frac{1}{w}\Sigma_1^\dagger\right)}\Sigma_1^2\right\} &= \frac{\mathrm{e}^{\frac{x}{2}(w+\frac{1}{w})}+\sigma^2\mathrm{e}^{\frac{x}{2}(w\sigma+\frac{1}{w}\sigma^2)}+\sigma\mathrm{e}^{\frac{x}{2}(w\sigma^2+\frac{1}{w}\sigma)}}{3}=\sum_{k\in\mathbf{Z}} I_{3k-2}(x)w^{3k-2}
\end{aligned} \tag{21}
$$

where $\sigma^{-1}=\sigma^2$ and $\sigma^{-2}=\sigma$.

A comment is in order. In the case of $n=2$ the generating matrix is

$$
\mathrm{e}^{\frac{x}{2}\left(w\sigma_1+\frac{1}{w}\sigma_1^\dagger\right)}=\mathrm{e}^{\frac{x}{2}\left(w+\frac{1}{w}\right)\sigma_1}=\cosh\left(\frac{x}{2}\left(w+\frac{1}{w}\right)\right)\mathbf{1}_2+\sinh\left(\frac{x}{2}\left(w+\frac{1}{w}\right)\right)\sigma_1
$$

because σ_1 is hermitian, so the situation becomes much easier.

From the lesson for the case of $n=3$, let us set up the general case. Let $\{\Sigma_1,\Sigma_3\}$ be

generalized Pauli matrices

$$\Sigma_1 = \begin{pmatrix} 0 & & & & & 1 \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix}, \qquad \Sigma_3 = \begin{pmatrix} 1 & & & & & \\ & \sigma & & & & \\ & & \sigma^2 & & & \\ & & & \ddots & & \\ & & & & \sigma^{n-2} & \\ & & & & & \sigma^{n-1} \end{pmatrix} \tag{22}$$

where σ is a primitive element $\sigma = \exp(\frac{2\pi i}{n})$ which satisfies

$$\sigma^n = 1, \quad \bar{\sigma} = \sigma^{n-1}, \quad 1 + \sigma + \cdots + \sigma^{n-1} = 0. \tag{23}$$

Then it is easy to see

$$\Sigma_1^n = \Sigma_3^n = \mathbf{1}_n, \quad \Sigma_1^\dagger = \Sigma_1^{n-1}, \quad \Sigma_3^\dagger = \Sigma_3^{n-1}, \quad \Sigma_3\Sigma_1 = \sigma\Sigma_1\Sigma_3. \tag{24}$$

If we define a Vandermonde matrix W based on σ as

$$W = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \sigma^{n-1} & \sigma^{2(n-1)} & \cdots & \sigma^{(n-2)(n-1)} & \sigma^{(n-1)^2} \\ 1 & \sigma^{n-2} & \sigma^{2(n-2)} & \cdots & \sigma^{(n-2)^2} & \sigma^{(n-1)(n-2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \sigma^2 & \sigma^4 & \cdots & \sigma^{2(n-2)} & \sigma^{2(n-1)} \\ 1 & \sigma & \sigma^2 & \cdots & \sigma^{n-2} & \sigma^{n-1} \end{pmatrix}, \tag{25}$$

then it is not difficult to see

$$\Sigma_1 = W\Sigma_3 W^\dagger = W\Sigma_3 W^{-1}. \tag{26}$$

That is, Σ_1 can be diagonalized by making use of W.

We set

$$c_j(x) = \sum_{k=0}^{\infty} \frac{x^{kn+j}}{(kn+j)!} \tag{27}$$

for $0 \leq j \leq n-1$. It is of course

$$\mathrm{e}^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{j=0}^{n-1} c_j(x)$$

and easy to see

$$\begin{aligned}
\mathrm{e}^{x\Sigma_1} &= c_0(x)\mathbf{1}_n + c_1(x)\Sigma_1 + c_2(x)\Sigma_1^2 + \cdots + c_{n-2}(x)\Sigma_1^{n-2} + c_{n-1}(x)\Sigma_1^{n-1} \\
&= \begin{pmatrix}
c_0(x) & c_{n-1}(x) & & \cdots & c_2(x) & c_1(x) \\
c_1(x) & c_0(x) & c_{n-1}(x) & \cdots & & c_2(x) \\
 & c_1(x) & c_0(x) & c_{n-1}(x) & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
c_{n-2}(x) & & \cdots & c_1(x) & c_0(x) & c_{n-1}(x) \\
c_{n-1}(x) & c_{n-2}(x) & \cdots & c_2(x) & c_1(x) & c_0(x)
\end{pmatrix}.
\end{aligned} \tag{28}$$

Let us look for the fundamental equation that $\{c_0(x), c_1(x), \cdots, c_{n-2}(x), c_{n-1}(x)\}$ satisfy. By use of (26)

$$\mathrm{e}^{x\Sigma_1} = \mathrm{e}^{xW\Sigma_3 W^\dagger} = W\mathrm{e}^{x\Sigma_3}W^\dagger$$

we have

$$\begin{aligned}
&\begin{vmatrix}
c_0(x) & c_{n-1}(x) & & \cdots & c_2(x) & c_1(x) \\
c_1(x) & c_0(x) & c_{n-1}(x) & \cdots & & c_2(x) \\
 & c_1(x) & c_0(x) & c_{n-1}(x) & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
c_{n-2}(x) & & \cdots & c_1(x) & c_0(x) & c_{n-1}(x) \\
c_{n-1}(x) & c_{n-2}(x) & \cdots & c_2(x) & c_1(x) & c_0(x)
\end{vmatrix} \\
=&\begin{vmatrix}
\mathrm{e}^{x} & & & & & \\
 & \mathrm{e}^{x\sigma} & & & & \\
 & & \mathrm{e}^{x\sigma^2} & & & \\
 & & & \ddots & & \\
 & & & & \mathrm{e}^{x\sigma^{n-2}} & \\
 & & & & & \mathrm{e}^{x\sigma^{n-1}}
\end{vmatrix} = \mathrm{e}^{x(1+\sigma+\sigma^2+\cdots+\sigma^{n-2}+\sigma^{n-1})} = \mathrm{e}^0 = 1
\end{aligned} \tag{29}$$

because W is unitary ($|W|=1$). For example

$$
\begin{aligned}
&n=2 \quad c_0^2(x)-c_1^2(x)=1 \quad (\Longleftarrow \ (6))\\
&n=3 \quad c_0^3(x)+c_1^3(x)+c_2^3(x)-3c_0(x)c_1(x)c_2(x)=1 \quad (\Longleftarrow \ (15))\\
&n=4 \quad c_0^4(x)-c_1^4(x)+c_2^4(x)-c_3^4(x)-2c_0^2(x)c_2^2(x)+2c_1^2(x)c_3^2(x)\\
&\qquad -4c_0^2(x)c_1(x)c_3(x)+4c_0(x)c_1^2(x)c_2(x)-4c_1(x)c_2^2(x)c_3(x)+4c_0(x)c_2(x)c_3^2(x)=1.
\end{aligned}
\tag{30}
$$

We call $\{c_0(x), c_1(x), \cdots, c_{n-1}(x)\}$ the super hyperbolic system.

The addition formulas are given by the simple equation

$$
\mathrm{e}^{x\Sigma_1}\mathrm{e}^{y\Sigma_1}=\mathrm{e}^{(x+y)\Sigma_1}
$$

and become

$$
c_j(x+y)=\sum_{k+l=j \pmod n} c_k(x)c_l(y) \quad \text{for} \quad 0\le j\le n-1. \tag{31}
$$

More explicitly,

$$
c_j(x+y)=c_0(x)c_j(y)+c_1(x)c_{j-1}(y)+\cdots+c_j(x)c_0(y)+c_{j+1}(x)c_{n-1}(y)+\cdots+c_{n-1}(x)c_{j+1}(y).
$$

The new relations are given by the simple equation

$$
\mathrm{e}^{x\Sigma_1}\mathrm{e}^{y\Sigma_1^\dagger}=\mathrm{e}^{x\Sigma_1+y\Sigma_1^\dagger}
$$

and become

$$
\sum_{k=0}^{j-1} c_k(x)c_{n-j+k}(y)+\sum_{k=j}^{n-1} c_k(x)c_{k-j}(y)=\frac{1}{n}\sum_{k=0}^{n-1}\sigma^{k(n-j)}\mathrm{e}^{x\sigma^k+y\sigma^{n-k}} \tag{32}
$$

for $0\le j\le n-1$.

The generating matrix of modified Bessel functions of integer order is given by

$$
\mathrm{e}^{\frac{x}{2}\left(w\Sigma_1+\frac{1}{w}\Sigma_1^\dagger\right)}=\sum_{k\in\mathbf{Z}} I_k(x)w^k\Sigma_1^k \tag{33}
$$

and from this we have

$$\frac{1}{n}\mathrm{tr}\left\{\mathrm{e}^{\frac{x}{2}\left(w\Sigma_1+\frac{1}{w}\Sigma_1^\dagger\right)}\Sigma_1^j\right\}=\frac{1}{n}\sum_{l=0}^{n-1}\sigma^{lj}\mathrm{e}^{\frac{x}{2}(w\sigma^l+\frac{1}{w}\sigma^{-l})}=\sum_{k\in\mathbf{Z}}I_{nk-j}(x)w^{nk-j} \tag{34}$$

for $0\le j\le n-1$.

The result in the case of $j=0$ is known in [6] and [7].

We want to take a (formal) limit $n\longrightarrow\infty$. That is, what is $\Sigma_1\longrightarrow ?$, $\Sigma_3\longrightarrow ?$ It is of course impossible to take a limit with this form. For that let us make a small change. We set $n=2N+1$ and

$$\widetilde{\Sigma}_1=\Sigma_1,\quad \widetilde{\Sigma}_3=\begin{pmatrix}\sigma^{-N} & & & & & & \\ & \ddots & & & & & \\ & & \sigma^{-1} & & & & \\ & & & 1 & & & \\ & & & & \sigma & & \\ & & & & & \ddots & \\ & & & & & & \sigma^N\end{pmatrix} \tag{35}$$

where $\sigma=\exp(\frac{2\pi i}{2N+1})$. Here we rewrite $\widetilde{\Sigma}_3$ as $\widetilde{\Sigma}_3=\exp(\frac{2\pi i}{2N+1}\widetilde{G})$ where

$$\widetilde{G}=\begin{pmatrix}-N & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & N\end{pmatrix}. \tag{36}$$

The commutator $[\widetilde{G}, \widetilde{\Sigma}_1]$ becomes

$$[\widetilde{G}, \widetilde{\Sigma}_1] = \begin{pmatrix} 0 & & & & & -2N \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & & & 1 \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix} \quad (\mathrm{mod}\ 2N+1).$$

That is, we have the relation

$$[\widetilde{G}, \widetilde{\Sigma}_1] = \widetilde{\Sigma}_1 \quad (\mathrm{mod}\ 2N+1). \tag{37}$$

In this stage, it may be better to write the (finite dimensional) Hilbert space as

$$\mathbf{C}^{2N+1} = \mathrm{Vect}_{\mathbf{C}}\{|-N\rangle, \cdots, |-1\rangle, |0\rangle, |1\rangle, \cdots, |N\rangle\}$$

because $\widetilde{G}|n\rangle = n|n\rangle$.

Now, if we take a formal limit $N \longrightarrow \infty$ then we have the fundamental relation

$$[G, W] = W \tag{38}$$

where

$$G = \begin{pmatrix} \ddots & & & & & \\ & -2 & & & & \\ & & -1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 2 & \\ & & & & & & \ddots \end{pmatrix}, \quad W = \begin{pmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{39}$$

where the notations $\{G, W\}$ in [4] were used. Note that G is a hermitian operator and W a unitary operator on the Hilbert space

$$\mathcal{L}^2(\mathbf{Z}) = \left\{ \sum_{n\in\mathbf{Z}} c_n|n\rangle \mid \sum_{n\in\mathbf{Z}} |c_n|^2 < \infty \right\}; \qquad W|n\rangle = |n+1\rangle, \quad G|n\rangle = n|n\rangle. \tag{40}$$

The relation (38) is just the fundamental one in quantum mechanice on the circle developed by Ohnuki and Kitakado [4].

A comment is in order. There is some freedom on the choice of G. That is, if we choose G like

$$G \longrightarrow G + \alpha\mathbf{1}, \quad 0 \le \alpha < 1$$

the relation (38) still holds. α is interpreted as a kind of abelian gauge induced in quantum mechanice on the circle.

Therefore it may be better to write the generators $\{G_\alpha \equiv G + \alpha\mathbf{1}, W\}$ in place of $\{G, W\}$ in [4]. We don't repeat the contents, so see [4] and its references.

Readers may find many interesting problems from the paper. For example, we can consider the **generating operator**

$$\mathrm{e}^{\frac{x}{2}\left(wW+\frac{1}{w}W^\dagger\right)}.$$

We leave some calculations to readers.

In this paper we developed the super hyperbolic structure for (all) finite quantum systems, and defined the generating matrix for the modified Bessel functions of integer order and obtained some interesting results. We also gave a connection to quantum mechanics on the circle by Ohnuki and Kitakado by taking a skillful limit.

Our motivation is to apply the development in the paper to qudit theory based on finite quantum systems, which will be reported in another paper.

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